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Topological Fluid Dynamics: Theory and Applications

Non-universal and non-singular asymptotics of interacting vortex filaments

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Abstract

We present a method for calculating the asymptotic shape of interacting vortex filaments in incompressible Euler flows using delay differential equations. Neglecting the filaments' core-size, the asymptotic shape of the filaments is self-similar up to logarithmic corrections, albeit non-universal. We demonstrate explicitly that the asymptotic geometry of the collapse of two interacting filaments depends on the pre-factor of the scaling law of their separation distance, the angle between the tangent vectors at their approaching tips, and the ratio of their circulations. We then explore the validity of the filament approximation in the limit of approaching the singularity. We show that a sufficiently fast stretching-rate to maintain this approximation is inconsistent with all collapse geometries. This suggests that a singular solution to the Euler equations based on stretching of vortex filaments is unlikely to exist for any initial conditions.

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1. Introduction

A fundamental open problem in mathematics and fluid dynamics is the global regularity of the three-dimensional incompressible Euler equations [1, 2, 3, 4, 5, 6]. One way to express this problem is whether a three-dimensional incompressible Euler flow with smooth initial conditions can develop a singularity with infinite vorticity in finite time. The Beale-Kato-Majda criterion [7] requires any solution with a finite-time blowup to have a divergence in the time integral of the maximum vorticity. For vortex tubes, this implies that at some parts the tubes must be stretched to the point of vanishing cross-section. Constructing flows that can stretch vortex lines thus becomes a promising approach in the search for a singularity. We focus on a subset of such flows: those involving interacting vortex filaments [8, 9, 10, 11, 12, 13, 14, 15, 16, 4].

Numerical searches for initial conditions of vortex filaments that can lead to a singularity have not been successful so far [3]. It seems that generic initial conditions are not promising candidates. A systematic search over all initial

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conditions is also not practically feasible; or worse, it is likely that singular solutions are unstable and not amenable to a numerical search.

We have proposed a different approach for searching: starting from the alleged singularity and working backwards [17]. Nothing precludes this a priori as the Euler equations are time-reversible. Our simplifying assumption is that infinitesimally close to the singularity (far away from the initial/boundary conditions), the dynamics follow scaling-laws set by dimensional analysis [18]. For vortex filaments, this means that the length scale characterising the *shape* of the filaments is $l(t) = \sqrt{\Gamma(t^* - t)}$, where Γ is the only dimensional parameter in the problem, the circulation, and t^*, the time of the alleged singularity. This vanishing length scale, characterising, for instance, the radius of curvature of the filaments and the inter-filament distance, captures the collapse of the filaments to a singularity. The core size, however, follows its own scaling law $\sigma(t) \sim (t^* - t)^{p/2}$. For a self-consistent collapse, where the filament approximation holds for all times, the core-size must vanish faster than the length-scale characterising the shape of the filaments. We must satisfy $p > 1$ to avoid core-deformation and have a self-consistent filament approximation. The different scaling of the core also implies that the solutions are not strictly self-similar [17]. Strictly self-similar solutions of finite-energy vortex filaments cannot be singular [19, 20]. See also [21, 22] for use of multiple length scales to characterise the collapse after the breakdown of the filament approximation.

Here, we analyse the asymptotic collapse geometry of two interacting vortex filaments. The collapse geometry is not universal and is characterised by a two-parameter family of solutions for fixed circulations. We explicitly demonstrate the dependence of the geometries on the pre-factor of the scaling law of the separation distance of the two filaments and the angle between the the tangents of the approaching tips. The asymptotic geometry also changes with the ratio of the circulation of the two filaments. In the last section, we relate the scaling exponent of the core to the collapse geometry, and argue that p vanishes in the asymptotic limit for all geometries. The culprit for this is the self-interaction term of the filaments, which has a logarithmic dependence on the core size. It is unlikely that a singular stretching of vortex filaments exists for any initial conditions. Sections 2, 3, 4, and 7 are succinct versions of the discussion in [17]; sections 5 and 6 expand upon these results, focussing on the non-universality of the asymptotic limit.

2. Filament approximation

Vortex filaments approximate the velocity field produced by vortex tubes, in which the vorticity distribution is limited to a tube of radius σ [23, 24]. When the radius of curvature of the filament is much larger than the core radius, the velocity field produced by each vortex filament is given by the regularised Biot-Savart law [23, 25],

$$\mathbf{v}(\mathbf{r_0}) = -\frac{\Gamma}{4\pi}\log(\frac{r_c}{\sigma})\kappa\hat{\mathbf{b}} - \frac{\Gamma}{4\pi}\int' \frac{(\mathbf{r_0} - \mathbf{r(s)}) \times \hat{\mathbf{t}}(s)}{|\mathbf{r_0} - \mathbf{r(s)}|^3}\, ds, \tag{1}$$

with Γ the circulation of the filament, $\mathbf{r}(s)$ the shape of the filament, $\mathbf{t}$ the tangent vector and $\mathbf{r_0}$ the location where the velocity field is measured. The approximation accurately captures the interaction of multiple filaments with each other, as long as the core radius of each filament is much smaller than the inter-filament distance. σ is the cut-off imposed to regularise the divergence in self-interaction; physically, it corresponds to the vortex core size. $\hat{\mathbf{b}} = \hat{\mathbf{t}} \times \hat{\mathbf{n}}$ is the binormal vector, and $\int'$ the regularised integral that runs along the non-local part of the filament. Note that the dynamics of the shape of the vortex filament depends very weakly (logarithmically) on the dynamics of the core (σ); hence the two problems are naturally decoupled.

The large fluid shears associated with colliding vortex filaments can cause dramatic changes to both the shape of the filament and the shape of the core. For the vortex filament approximation to remain an accurate description of a collision, the core radii of the vortex filaments must remain smaller than the length-scales characterising the shape of the filaments, uniformly in time; this means that the radius of the tubes must shrink more quickly than the distance between colliding filaments.

The filament core radius evolves to satisfy volume conservation. If s measures arc length along a filament, with the original filament parameterised by α, then s_α measures the stretching of a filament. We then have

$$\sigma^2 = \frac{\sigma_0^2}{s_\alpha}. \tag{2}$$

In general, if the dynamics of the vortex core is decomposed into a local coordinate system,

$$\frac{d\mathbf{r}}{dt} = W\hat{\mathbf{t}} + U\hat{\mathbf{n}} + V\hat{\mathbf{b}}, \tag{3}$$

where $\mathbf{n}, \mathbf{t}, \mathbf{b}$ are respectively the normal, tangent and binormal vectors, then the stretching of the filament evolves according to

$$\frac{ds_\alpha}{dt} = (\frac{d\mathbf{v}}{ds} \cdot \hat{\mathbf{t}})s_\alpha = \frac{dW}{ds}s_\alpha - U\kappa s_\alpha. \tag{4}$$

The first term on the right hand side of Eq. (4) is the stretching of the filament due to local shear, whereas the second term is due to motion in the normal direction. Since the circulation is constant, the local vorticity in the vortex filament is given by $\omega = \Gamma(\pi\sigma^2)^{-1} \propto \Gamma s_\alpha$. Hence, a diverging vorticity is equivalent to a diverging s_α.

3. The asymptotic ansatz

The key assumption in the asymptotic limit is that the shape of the filaments follow the scaling law set by the dimension of circulation to leading order, coupled to a core that follows its own scaling law. There are two length scales – one characterising filament shape and the other the size of the core, whose scaling exponents are coupled through the governing equations. We want to know if the asymptotic dynamics can self-consistently stretch the core to generate a singularity.

From dimensional analysis (see, e.g. [26]), to leading-order, the characteristic length scales governing the filament shapes are $\ell_i(t) = \sqrt{|\Gamma_i|(t^* - t)}$, where $i = 1, 2$ denotes filament one or two with circulation Γ_i, and t^* the time of singularity. Throughout the remainder of this paper, we write all the equations for only the first filament $i = 1$, with the second filament obeying the complementary equation. The shapes of the filament then take the form

$$\mathbf{r}_1(s, t) = \ell_1(t)\mathbf{G}_1(\eta), \tag{5}$$

where $\eta = s/\ell_1(t)$, and s measures arc length along the filament. In general, $\mathbf{G}$ can have explicit time-dependence; we neglect this now, but return to it at the end of this paper. Substituting the above similarity ansatz into Eq. 1 gives a set of coupled ordinary integro-differential equations for the shapes of the filament.

These equations are difficult to solve (see section below), but we can still use the asymptotic solution Eq. 5 to derive a condition for the self-consistency of the collapse of the core. Substituting the asymptotic ansatz into Eq. 4: The normal and tangential self-similar velocity components obey $U = u(\eta)\sqrt{|\Gamma_i|/(t^* - t)}$ and $W = w(\eta)\sqrt{|\Gamma_i|/(t^* - t)}$, whereas the curvature of the filament is given by $\kappa = k(\eta)/\sqrt{|\Gamma_i|(t^* - t)}$; hence, $U\kappa = u(\eta)k(\eta)/(t^* - t)$ and $W_s = w'(\eta)/(t^* - t)$. Here, $u(\eta)$, $k(\eta)$, and $w(\eta)$ are location dependent pre-factors, given by the solution of asymptotic equations. Putting this together in Eq. 4, we obtain that

$$\frac{ds_\alpha}{dt} = \frac{w'(\eta) - u(\eta)k(\eta)}{t^* - t}s_\alpha. \tag{6}$$

Hence, the stretching rate of the filament obeys a power law $s_\alpha \sim (t^* - t)^{-p(\eta)}$, with the position-dependent exponent $p(\eta)$ given by the asymptotic solution! Eq. 2 then implies that the filament radius vanishes according to $\sigma \sim (t^* - t)^{p(\eta)/2}$. The observation that the pre-factors in the asymptotic solutions for outer filaments control the power-law exponents for the core collapse was anticipated by Moffatt [27], who studied the behaviour of inviscid vortex filaments under power-law diverging strains.

We have therefore arrived at a criterion for self-consistency of the collapsing filament solution: self-consistency requires that the filament radius decrease faster than the filament separation distance, or $p > 1$. Indeed, since the vorticity scales with s_α, this is a realisation of the Beale-Kato-Majda criterion [7] for vortex filaments. $p = 1$ is not allowed, since it would imply a solution where all length-scales follow the same scaling law, namely $\sim \sqrt{t^* - t}$. Such finite-energy 'strictly' self-similar solutions cannot be singular [19]. The condition $p > 1$ also satisfies the geometrical constraints of Constantin, Fefferman & Majda [28] and bounds of Deng, Hou, & Yu [29] for a singularity.

4. Analysis

Substituting the similarity ansatz (Eq. 5) into Eq. 1 results in an equation for the shape of the filaments,

$$\mathbf{G}_1 - \eta \mathbf{G}_1' \approx \alpha \frac{\Gamma_1}{|\Gamma_1|} \mathbf{G}_1' \times \mathbf{G}_1'' + \frac{\Gamma_2 \sqrt{|\Gamma_2|}}{2\pi \sqrt{|\Gamma_1|}} \int \frac{\left(\sqrt{|\Gamma_1|}\mathbf{G}_1(\eta) - \sqrt{|\Gamma_2|}\mathbf{G}_2(\zeta)\right) \times \mathbf{G}_2'(\zeta)}{|\sqrt{|\Gamma_1|}\mathbf{G}_1(\eta) - \sqrt{|\Gamma_2|}\mathbf{G}_2(\zeta)|^3} d\zeta, \tag{7}$$

where $\alpha = \frac{1}{2\pi} \ln\left(\frac{r_c}{\sigma}\right)$ is the self-interaction term, and the integral, the non-local contribution of the Biot-Savart kernel. The approximate equality indicates that the two sides of the above expression are only equal modulo the tangential component of the velocity along the filament. Since the tangential component of the velocity does not change the shape of the filaments, we can still use Eq. 7 to compute the asymptotic collapse geometry.

The self-interaction term does not contribute to stretching since it points along the binormal direction. The only term that needs to be considered is the non-local contribution. To proceed, we first derive an explicit expression for the highest-order derivative term $\mathbf{G}_1''$ from Eq. 7. However, as noted, Eq. 7 is only an equality modulo the component of the velocity field tangent to the filaments. To get around this nuisance, we project out the tangential component by taking the cross product of both sides of Eq. 7 with $\mathbf{G}_1'$. The first term on the right hand side simplifies further using the vector identity, $\mathbf{G}_1' \times \mathbf{G}_1' \times \mathbf{G}_1'' = \mathbf{G}_1'(\mathbf{G}_1' \cdot \mathbf{G}_1'') - \mathbf{G}_1''(\mathbf{G}_1' \cdot \mathbf{G}_1') = -\mathbf{G}_1''$, where in the last step we have used $|\mathbf{G}_1'| = 1$, or equivalently $\mathbf{G}_1' \cdot \mathbf{G}_1'' = 0$.

Following the above procedure, $\mathbf{G}_1''$ is given by,

$$\mathbf{G}_1''(\eta) = -\frac{|\Gamma_1|}{\alpha \Gamma_1} \mathbf{G}_1'(\eta) \times \left(\mathbf{G}_1(\eta) - \frac{\Gamma_2 \sqrt{|\Gamma_2|}}{2\pi \sqrt{|\Gamma_1|}} \int \frac{\left(\sqrt{|\Gamma_1|}\mathbf{G}_1(\eta) - \sqrt{|\Gamma_2|}\mathbf{G}_2(\zeta)\right) \times \mathbf{G}_2'(\zeta)}{|\sqrt{|\Gamma_1|}\mathbf{G}_1(\eta) - \sqrt{|\Gamma_2|}\mathbf{G}_2(\zeta)|^3} d\zeta \right). \tag{8}$$

The integration runs along the length of the second filament and the non-local portion of the first filament, which can be thought of as another filament.

The shape of the filaments (asymptotic solution) has an explicit time dependence through the parameter $\alpha = (2\pi)^{-1} log(r_c/\sigma)$. If we assume that $r_c/\sigma \to \infty$ as $t \to t^*$, so that the asymptotic solution is self-consistent, then $\alpha \to \infty$ as $t \to t^*$. Eq. 8 then implies that in the limit of approaching the singularity, $\mathbf{G}'' \to 0$; the filament curvature in similarity space asymptotically vanishes. The curvature in real space $|\mathbf{G}''|/l_1$ still diverges, however, more slowly than $1/\sqrt{t^* - t}$. This assertion is only valid assuming that the non-local integral does not compensate for the growth in α. The non-local contributions is bounded in the limit $t \to t^*$, since it corresponds to the Biot-Savart integral over filaments with asymptotically fixed shape (time-independent integrand). For a discussion on non-self-similar solutions (where the integrand is time-dependent) see [17]. Time-dependence of the limits of this integral can not compensate for any form of divergence in α, since to properly match any solution of Eq. 8 to an outer solution, the asymptotic limit $\eta \to \infty$ must satisfy $\mathbf{G} \sim \eta$. Contribution of a straight line to the Biot-Savart integral even when extended to infinity is always finite.

Given the vanishing curvature, it is possible to replace the interaction integral in Eq. 8 with a (non-integral) interaction term of straight filaments. Note that in the limit of approaching the singularity this approximation becomes exact as the filaments become straight lines; the corrections to the approximation are of the order $1/\alpha$ and vanish for a self-consistent singularity.

5. Delay differential equations

Under the straight filament approximation the velocity induced at point R by a vortex filament is

$$\mathbf{v}(\mathbf{r_0}) = -\frac{\Gamma}{4\pi} \log\left(\frac{r_c}{\sigma}\right) \kappa \hat{\mathbf{b}} - \frac{\Gamma}{2\pi} \frac{(\mathbf{r_0} - \mathbf{r}(\mathbf{s})) \times \hat{\mathbf{t}}(s)}{|\mathbf{r_0} - \mathbf{r}(\mathbf{s})|^2}. \tag{9}$$

where $\mathbf{r}(\mathbf{s})$ is the closest point on the filament to point $\mathbf{r_0}$. For well-behaved geometries, this is equivalent to

$$\frac{d|\mathbf{r_0} - \mathbf{r}(\mathbf{s})|}{ds} = 0. \tag{10}$$

The arc length parameter η is used to denote the distance in similarity space along the filament from the tip. We define two other arc length parameters η_1 and η_2 to designate the corresponding nearest points on the other filament. This means that on filament 1, $\mathbf{G}_1(\eta)$ is closest to $\mathbf{G}_2(\eta_2)$; and on filament 2, $\mathbf{G}_2(\eta)$ is closest to $\mathbf{G}_1(\eta_1)$. Equivalently,

$$\frac{d}{d\eta_2}|\sqrt{|\Gamma_2|}\mathbf{G}_2(\eta_2) - \sqrt{|\Gamma_1|}\mathbf{G}_1(\eta)|^2 = 0. \tag{11}$$

For well behaved geometries (and fixed η), this is equivalent to,

$$\sqrt{|\Gamma_2|}\mathbf{G}_2'(\eta_2)\cdot(\sqrt{|\Gamma_2|}\mathbf{G}_2(\eta_2) - \sqrt{|\Gamma_1|}\mathbf{G}_1(\eta)) = 0. \tag{12}$$

Geometrically, the above equation tells us that the line connecting a point on filament 1 to the nearest point on filament 2 should be perpendicular to the tangential component of filament 2 at that point. η_2 is clearly a function of η. Its differential dependence on η is given by,

$$\frac{d\eta_2}{d\eta} = \frac{\sqrt{|\Gamma_1|}\mathbf{G}_2'(\eta_2)\cdot\mathbf{G}_1'(\eta)}{\mathbf{G}_2''(\eta_2)\cdot\left(\sqrt{|\Gamma_2|}\mathbf{G}_2(\eta_2) - \sqrt{|\Gamma_1|}\mathbf{G}_1(\eta)\right) + \sqrt{|\Gamma_2|}|\mathbf{G}_2'(\eta_2)|^2} \tag{13}$$

Applying the straight filament approximation to Eq.8 gives

$$\mathbf{G}_1''(\eta) = -\frac{|\Gamma_1|}{\alpha_1\Gamma_1}\mathbf{G}_1'(\eta)\times\left(\mathbf{G}_1(\eta) - \frac{\Gamma_2}{\pi\sqrt{|\Gamma_1|}}\frac{\left(\sqrt{|\Gamma_1|}\mathbf{G}_1(\eta) - \sqrt{|\Gamma_2|}\mathbf{G}_2(\eta_2)\right)\times\mathbf{G}_2'(\eta_2)}{|\sqrt{|\Gamma_1|}\mathbf{G}_1(\eta) - \sqrt{|\Gamma_2|}\mathbf{G}_2(\eta_2)|^2}\right). \tag{14}$$

To evaluate $\mathbf{G}_1''(\eta)$ we need the position (and the tangent) of the point $\mathbf{G}_2(\eta_2)$ on the second filament closest to $\mathbf{G}_1(\eta)$. We have derived an expression for the evolution of η_2 (Eq.13). To have a closed set of ordinary differential equations that can be solved numerically using iterative methods, we need to go to the higher order differential of filament shape $\mathbf{G}_1'''$. Differentiating the last equation with respect to η gives,

$$\begin{aligned}\mathbf{G}_1'''(\eta) = &-\frac{|\Gamma_1|}{\alpha_1\Gamma_1}\mathbf{G}_1''(\eta)\times\left(\mathbf{G}_1(\eta) - \frac{\Gamma_2}{\pi\sqrt{|\Gamma_1|}}\frac{\left(\sqrt{|\Gamma_1|}\mathbf{G}_1(\eta) - \sqrt{|\Gamma_2|}\mathbf{G}_2(\eta_2)\right)\times\mathbf{G}_2'(\eta_2)}{|\sqrt{|\Gamma_1|}\mathbf{G}_1(\eta) - \sqrt{|\Gamma_2|}\mathbf{G}_2(\eta_2)|^2}\right)\\ &-\frac{|\Gamma_1|}{\alpha_1\Gamma_1}\mathbf{G}_1'(\eta)\times\Bigg(\mathbf{G}_1'(\eta) - \frac{\Gamma_2}{\pi\sqrt{|\Gamma_1|}}\frac{\left(\sqrt{|\Gamma_1|}\mathbf{G}_1'(\eta) - \sqrt{|\Gamma_2|}\mathbf{G}_2'(\eta_2)\frac{d\eta_2}{d\eta}\right)\times\mathbf{G}_2'(\eta_2)}{|\sqrt{|\Gamma_1|}\mathbf{G}_1(\eta) - \sqrt{|\Gamma_2|}\mathbf{G}_2(\eta_2)|^2}\\ &-\frac{\Gamma_2}{\pi\sqrt{|\Gamma_1|}}\frac{\left(\sqrt{|\Gamma_1|}\mathbf{G}_1(\eta) - \sqrt{|\Gamma_2|}\mathbf{G}_2(\eta_2)\right)\times\mathbf{G}_2''(\eta_2)\frac{d\eta_2}{d\eta}}{|\sqrt{|\Gamma_1|}\mathbf{G}_1(\eta) - \sqrt{|\Gamma_2|}G_2(\eta_2)|^2}\\ &+\frac{2\Gamma_2}{\pi\sqrt{|\Gamma_1|}}\frac{\left(\sqrt{|\Gamma_1|}\mathbf{G}_1(\eta) - \sqrt{|\Gamma_2|}\mathbf{G}_2(\eta_2)\right)\times\mathbf{G}_2'(\eta_2)}{|\sqrt{|\Gamma_1|}\mathbf{G}_1(\eta) - \sqrt{|\Gamma_2|}\mathbf{G}_2(\eta_2)|^4}\\ &\left(\sqrt{|\Gamma_1|}\mathbf{G}_1'(\eta) - \sqrt{|\Gamma_2|}\mathbf{G}_2'(\eta_2)\frac{d\eta_2}{d\eta}\right)\cdot\left(\sqrt{|\Gamma_1|}\mathbf{G}_1(\eta) - \sqrt{|\Gamma_2|}G_2(\eta_2)\right)\Bigg)\end{aligned} \tag{15}$$

Eq.13 and Eq. 16, alongside two equivalent equations for the shape of the second filament $\mathbf{G}_2$, form a set of first-order coupled delay ordinary differential equations (ODEs) in 20 variables. The initial conditions that need to be specified are the values of $\mathbf{G}_1$, $\mathbf{G}_2$, $\mathbf{G}_1'$, $\mathbf{G}_2'$ at the tip of the filaments $\eta = 0$. $\mathbf{G}_1''$, and $\mathbf{G}_2''$ (using Eq. 14) are then computed and form the remaining initial conditions. We also impose that the tips are reciprocally the closest two points on the filaments $\eta_1(0) = 0$ and $\eta_2(0) = 0$. The coupled ODEs are solved using an explicit iterative method.

Using the symmetries in the problem, we can enumerate the number of independent degrees of freedom in the initial conditions that can give rise to distinct collapse geometries. We assume that the two filaments have equal magnitudes of circulation, $\Gamma_1 = -\Gamma_2 = \Gamma$. There are two independent degrees of freedom in the initial conditions,

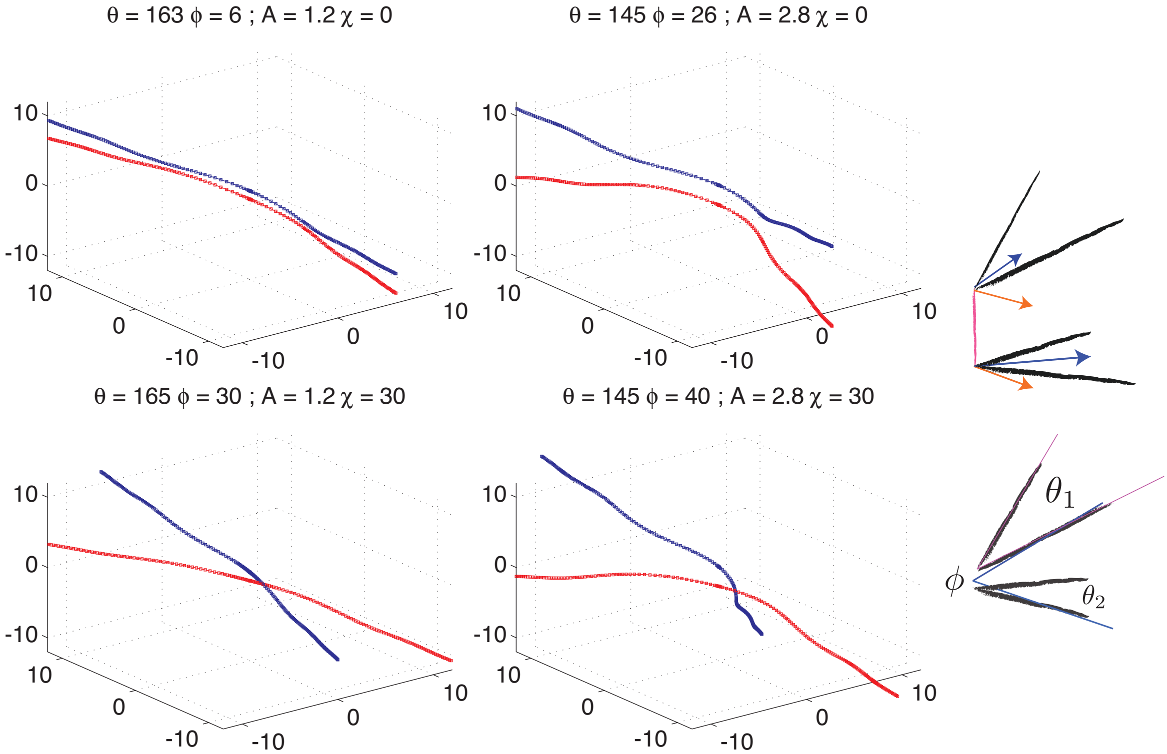

Fig. 1. Collapse geometries. Solution of the delay ODEs, $\mathbf{G}_{1,2}(\eta)$, for different values of parameters A and χ (in degrees); $\alpha = 25$. $\Gamma_1 = -\Gamma_2 = 1$. The shape of the filaments, quantified by the opening angles θ and ϕ, varies for different values of A and χ. The definition of opening angles is shown on the bottom right. The tangent vectors at the tips of the filaments are depicted as orange vectors (top right).

generating a two-parameter family of solutions. One parameter corresponds to the separation distance of the tips of the filaments in similarity space, $A = |\mathbf{G}_1(0) - \mathbf{G}_2(0)|$. When scaled back to real space, A is the pre-factor of the scaling law $(A\sqrt{\Gamma(t^* - t)})$ of the inter-filament separation distance. The other parameter is the angle χ between the tangent vectors at the tip of the filaments, $\mathbf{G}_1'(0)$ and $\mathbf{G}_2'(0)$. Since the tips are by definition the two closest points on the filaments, $\mathbf{G}_{1,2}'(0) \cdot (\mathbf{G}_1(0) - \mathbf{G}_2(0)) = 0$. The direction of the tip tangent vectors can be characterized using only angle χ up to a trivial global rotation.

6. Non-universality of similarity solutions

To demonstrate the non-universality of the collapse geometry, we have solved the similarity delay ODEs using a four-stage, fourth-order explicit Runge-Kutta iterative method [30], for various values of the two parameters A and χ. The shape of the filaments is quantified using the geometry of the resulting 'tent', as characterized by the opening angles $\theta_{1,2}$ and $\phi_{1,2}$ (see Fig.1). Because the filaments are assumed to have the same circulation magnitude, for all the computed geometries, $\theta_1 = \theta_2 = \theta$ and $\phi_1 = \phi_2 = \phi$ (for the case with different circulations see below).

As demonstrated in Fig.1, the choice of parameter A modifies the geometry of the solution as reflected in the different opening angles. Similarly, changing the orientation of the tangent vectors, angle χ, also results in a different asymptotic geometry. Fig.2 depicts a systematic sweep of the two parameters. Changing A significantly modifies the intra-filament opening angle θ, whereas, changing χ mostly modifies the inter-filament angle ϕ. It is also possible to explore asymptotic collapse geometries where the magnitude of circulation is not the same for the two filaments. With different circulations, the shape of the filaments are not identical – i.e. as characterised by the intra-filament opening angles $\theta_1 \neq \theta_2$, resulting in an asymmetric collapse geometry. Fig.3 depicts the asymptotic solutions for a variety of Γ_2 values for a fixed circulation in the first filament, $\Gamma_1 = -1$. For some values of asymmetric circulations,

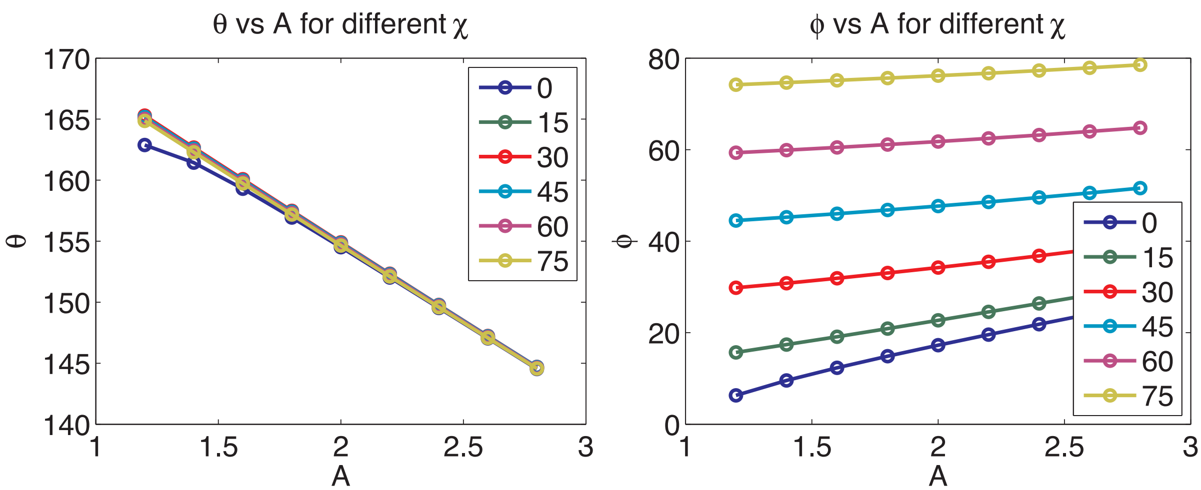

Fig. 2. Non-universality of the asymptotic solutions. Left: Intra-filament opening angle (θ) as a function of A, the pre-factor of the scaling of the filament separation distance, for different values of χ (in degrees). θ does not show a strong dependence on χ. Right: Inter-filament opening angle (ϕ) as a function of A for same set of χ values.

one filament crosses the other's opening plane (the plane spanned by the filaments' arms emanating from its tip); this results in the non-monotonous behaviour of ϕ_1 with increasing Γ_2 seen in Fig.3.

Since the asymptotic geometry is not unique and dependent on the initial conditions in similarity space (or equivalently pre-factors of scaling laws in real space), the collapse geometry is potentially non-universal. Of course, this is a necessary but not sufficient condition for non-universality. We need to show that different asymptotic geometries correspond to well-defined initial conditions in real space. For instance, it is conceivable that the asymptotic filament shapes with non-zero χ can not be matched to any real space solution of two interacting filaments (for a general discussion on matching see [31]). Nevertheless, recent numerical analysis of reconnection of vortex rings and lines [32, 33] (primarily focussing on Gross-Pitaevskii equations with relevance in quantum vortex reconnections [34, 35]) have demonstrated dependence of scaling pre-factors on the initial conditions and a variety of 'tent' geometries prior to reconnection. This is contrary to the claims in [36] that suggests a universal geometry for reconnection of two vortex filaments.

Non-universality of the asymptotic collapse is encouraging for finding a singularity. If one particular asymptotic geometry corresponds to a self-consistent singularity, then a search for the corresponding initial conditions is justified. However, as we show below, the entire family of filament shapes in similarity space will eventually succumb to core-deformation, precluding the possibility of singular stretching.

7. Absence of singularity

By substituting the scaling ansatz of the core into the stretching equation (Eq.6), we can solve for the scaling exponent p. Stretching is caused by the non-local contribution to the velocity, since local self-induced velocity points in binormal direction and does not stretch. The stretching exponent p is given by:

$$p(\eta) = \frac{\sqrt{t^* - t}}{\sqrt{|\Gamma_1|}} \left(\frac{d\mathbf{v}_1}{d\eta} \cdot \mathbf{G}_1'(\eta) \right). \tag{16}$$

Evaluating this expression at $\eta = 0$, which by definition corresponds to the tip of the collapsing filaments (see above), gives,

$$p_1(\eta = 0) = -\frac{\Gamma_2}{2\pi\sqrt{|\Gamma_1|}} \frac{\sqrt{|\Gamma_1|}\mathbf{G}_1(0) - \sqrt{|\Gamma_2|}\mathbf{G}_2(0)}{|\sqrt{|\Gamma_1|}\mathbf{G}_1(0) - \sqrt{|\Gamma_2|}\mathbf{G}_2(0)|^2} \cdot \left(\mathbf{G}_2''(0) \frac{d\eta_2}{d\eta} \times \mathbf{G}_1'(0) \right). \tag{17}$$

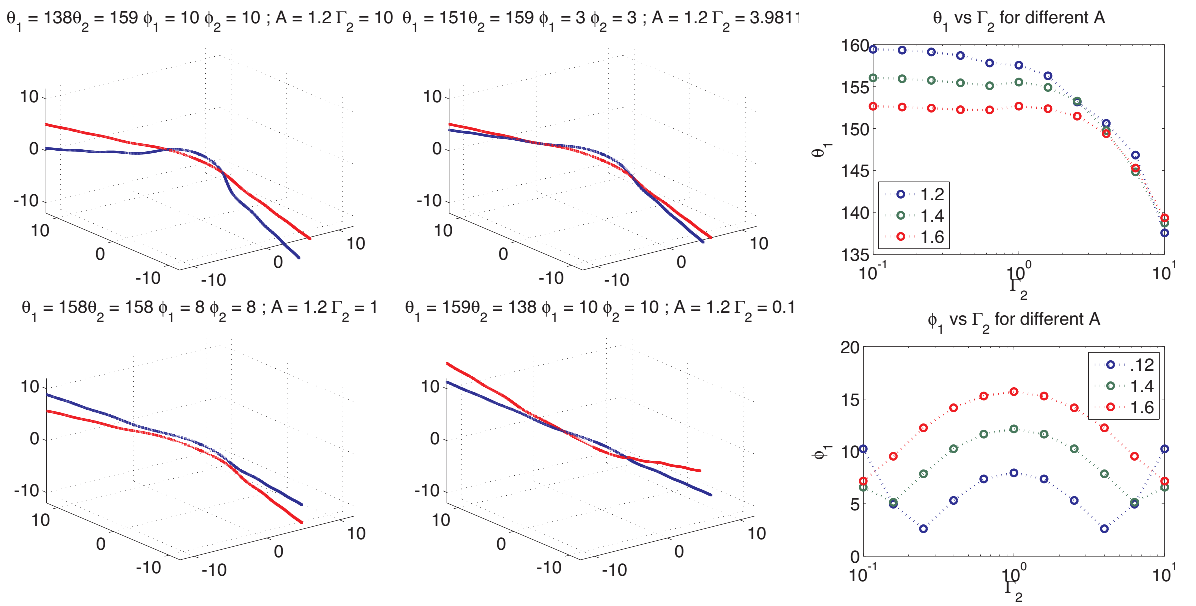

Fig. 3. Asymmetrical circulations. Left: Asymptotic collapse geometries for $\Gamma_2 = 10, 3.981, 1, 0.1$; for all cases, $\Gamma_1 = -1$, $\alpha = 15$, $A = 1.2$, and $\chi = 0$. With different circulations, the filaments have different shapes, i.e. $\theta_1 \neq \theta_2$, resulting in asymmetrical collapse geometry. The asymptotic shape as quantified by the opening angles varies with the changing Γ_2. Right: θ_1 and ϕ_1 opening angles for a systematic sweep of Γ_2 for $A = 1.2, 1.4, 1.6$, $\alpha = 15$, and $\chi = 0$.

The most important feature of this result is that $p(\eta = 0)$ *vanishes* in the limit of $\tau \to \infty$ because $\mathbf{G}''$ vanishes in this limit. We assumed $p > 1$ for a self-consistent singular collapse, but computed $p \to 0$ in the limit of approaching the singularity, resulting in a contradiction. This shows that singularities of pairs of vortex filaments can not happen. We remark that the term $\frac{d\eta_2}{d\eta}$, capturing the asymmetry between the shape of the two filaments, remains regular because the circulations $\Gamma_{1,2}$ are always finite (see Eq. 13). The logarithmic corrections to the scaling law imply that the self-similar solution is not strictly valid; there are dynamics in $log(t^* - t)$. Although it is unlikely that even non-self-similar singular solution can exist involving stretching of vortex filaments (for details see [17]).

The essential reason for lack of any solution with singular stretching is that the rather slow (logarithmic) flattening out of the filaments eventually overcomes any clever tricks with filament shapes that can be incorporated using initial conditions or unequal circulations.

8. Summary

By imposing a self-similar ansatz on the length scales characterising the *shape* of vortex filaments (not their core size), we were able to calculate the filaments' collapse geometry in the asymptotic limit of approaching a singularity. The collapse geometries are not universal; for the case of two filaments, they form a family of solutions parametrized by the pre-factor of the scaling of the inter-filament distance and the angle between the tangents at the filaments' tips. We then argued that for all asymptotic geometries, the logarithmic coupling between the core size and the self-induced velocity prohibits the core from shrinking fast enough to maintain the filament approximation all the way to a singularity. In real space, this is manifested as a logarithmic correction to the scaling of the radius of curvature of the filaments (which is considerably difficult to detect numerically): the separation distance of the filaments vanishes faster than their radius of curvature by a factor of $-log(t^* - t)$, eventually resulting in core-deformation. Our starting assumption of existence of a singularity through stretching of vortex filaments is not self-consistent. A generalisation of this argument to multiple filaments suggests that singular stretching of vortex filaments is not possible for any set of initial conditions.

It is worthwhile to extend this methodology to other promising mechanism for generating a singularity in the Euler equations. Arbitrary close to an alleged singularity, a vanishing length scale should provide simplified dynamics, for instance self-similarity. Working backwards from the singularity, the simplified dynamics can be assumed and then checked for self-consistency. If the asymptotic limit proves self-consistent (unlike the above case), the solution must be matched back to the regular dynamics and a suitable set of initial conditions.

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References

[1] Leray J. On the motion of a viscous liquid filling space. *Acta Mathematica*. 1934; 63:193–248.
[2] Majda A, Bertozzi AL. *Vorticity and Incompressible Flow*. Cambridge University Press, Cambridge; 2001.
[3] Gibbon J. The three-dimensional euler equations: Where do we stand? *Physica D: Nonlinear Phenomena*. 2008; 237:1894–1904.
[4] Childress S. Growth of anti-parallel vorticity in euler flows. *Physica D: Nonlinear Phenomena*. 2008; 237:1921–1925.
[5] Constantin P. Singular, weak and absent: Solutions of the euler equations. *Physica D: Nonlinear Phenomena*. 2008; 237:1926–1931.
[6] Pomeau Y, Sciamarella D. An unfinished tale of nonlinear PDEs: Do solutions of 3D incompressible Euler equations blow-up in finite time? *Physica D: Nonlin. Phen.* 2005; 205:215–221.
[7] Beale J, Kato T, Majda A. Remarks on the breakdown of smooth solutions for the 3-d euler equations. *Communications in Mathematical Physics*. 1984; 94:61–66.
[8] Morf R, Orszag S, Frisch U. Spontaneous singularity in three-dimensional inviscid, incompressible flow. *Physical Review Letters*. 1980; 44:572–575.
[9] Siggia E. Collapse and amplification of a vortex filament. *Physics of Fluids*. 1985; 28:794–805.
[10] Pumir A, Siggia E. Vortex dynamics and the existence of solutions to the navier–stokes equations. *Physics of Fluids*. 1987; 30:1606–1626.
[11] Pumir A, Siggia E. Collapsing solutions to the 3-d euler equations. *Phys Fluids A-Fluid*. 1990; 2:220–241.
[12] Klein R, Majda A. An asymptotic theory for the nonlinear instability of antiparallel pairs of vortex filaments. *Physics of Fluids A: Fluid Dynamics*. 1993; 5:369–379.
[13] Kerr R. Evidence for a singularity of the threedimensional, incompressible euler equations. *Physics of Fluids A: Fluid Dynamics*. 1993; 5:1725–1746.
[14] Klein R, Majda A, Damodaran K. Simplified equations for the interaction of nearly parallel vortex filaments. *Journal of Fluid Mechanics*. 1995; 288:201–248.
[15] Pelz R. Locally self-similar, finite-time collapse in a high-symmetry vortex filament model. *Phys Rev E*. 1997; 55:1617–1626.
[16] Kimura Y. Self-similar collapse of 2D and 3D vortex filament models. *Theor. Comput. Fluid Dyn.* 2010; 24:389-394.
[17] Hormoz S, Brenner MP. Absence of singular stretching of vortex filaments. *Journal of Fluid Mechanics*. 2012; 707:191-204. DOI: http://dx.doi.org/10.1017/jfm.2012.270.
[18] Barenblatt GI. *Scaling, Self-similarity and Intermediate Asymptotics* Cambridge University Press, Cambridge; 1996.
[19] Chae D. Nonexistence of self-similar singularities for the 3d incompressible euler equations. *Communications in Mathematical Physics*. 2007; 273:203–215, 10.1007/s00220-007-0249-8.
[20] Chae D. On the generalized self-similar singularities for the Euler and the Navier-Stokes equations. *J. Funct. Anal.* 2010; 258:2865-2883.
[21] Kerr RM. Velocity and scaling of collapsing Euler vortices. *Phys. Fluids*. 2005; 17:075103.
[22] Bustamante MD, Kerr RM. 3D Euler about a 2D symmetry plane. *Physica D: Nonlinear Phenomena*. 2008; 237:1912–1920.
[23] Schwarz KW. Three-dimensional vortex dynamics in superfluid ^{4}He: Line-line and line-boundary interactions. *Phys. Rev. B*. 1985; 31:57825804.
[24] Saffman PG. *Vortex Dynamics*. Cambridge: Cambridge Univ. Press; 1992.
[25] Saffman P, Baker G. Vortex interactions. *Annu Rev Fluid Mech*. 1979; 11:95–122.
[26] Gutierrez S, Rivas J, , Vega L. Formation of singularities and self-similar vortex motion under the localized induction approximation. *Communications in Partial Differential Equations*. 2003; 28:927–968.
[27] Moffatt HK.The interaction of skewed vortex pairs: a model for blow-up of the navier–stokes equations. *Journal of Fluid Mechanics*. 2000; 49:51–68.
[28] Constantin P, Fefferman C, Majda A. Geometric constraints on potentially singular solutions for the 3-d euler equations. *Commun Partial Differential Equations*. 1996; 21:559–571.
[29] Deng J, Hou T, Yu X. Improved geometric conditions for non-blowup of the 3d incompressible euler equation. *Comm in Partial Differential Equations*. 2006; 31:293–306.
[30] Shampine LF. Solving ODEs and DDEs with Residual Control. *Applied Numerical Mathematics*. 2005; 52:113–127.
[31] Eggers J, Fontelos A. The role of self-similarity in singularities of partial differential equations. *Nonlinearity*. 2009; 22:R1–R44.
[32] Tebbs R, Youd AJ, Barenghi CF. The Approach to Vortex Reconnection. *Journal of Low Temperature Physics*. 2011;162:314–321.

[33] Zuccher S, Caliari M, Baggaley AW, Barenghi CF. Quantum vortex reconnections. *arXiv:1206.2498v1*
[34] Paoletti MS, Fisher ME, Sreenivasan KR, Lathrop DP. Velocity statistics distinguish quantum turbulence from classical turbulence. *Phys Rev Lett*. 2008; 101:154501.
[35] Paoletti MS, Fisher ME, Lathrop DP. Reconnection dynamics for quantized vortices. *Physica D: Nonlinear Phenomena*. 2010; 239:1367–1377.
[36] de Waele A, Aarts R. Route to vortex reconnection. *Phys Rev Lett*. 1994; 72:482–485.